Permutations with few inversions are locally uniform

David Bevan[†]

Abstract

We prove that permutations with few inversions exhibit a local–global dichotomy in the following sense. Suppose $\boldsymbol{\sigma}$ is a permutation chosen uniformly at random from the set of all permutations of $[n]$ with exactly $m = m(n) \ll n^2$ inversions. If $i < j$ are chosen uniformly at random from $[n]$, then $\boldsymbol{\sigma}(i) < \boldsymbol{\sigma}(j)$ asymptotically almost surely. However, if i and j are chosen so that $j - i \ll m/n$, and $m \ll n^2/\log^2 n$, then $\lim_{n\to\infty} \mathbb{P}[\boldsymbol{\sigma}(i) < \boldsymbol{\sigma}(j)] = \frac{1}{2}$. Moreover, if $k = k(n) \ll \sqrt{m/n}$, then the restriction of $\boldsymbol{\sigma}$ to a random k-point interval is asymptotically uniformly distributed over $\mathcal{S}_k$. Thus, knowledge of the local structure of $\boldsymbol{\sigma}$ reveals nothing about its global form. We establish that $\sqrt{m/n}$ is the threshold for local uniformity and m/n the threshold for inversions, and determine the behaviour in the critical windows.

1 Introduction

We consider a permutation σ of length n (an n-*permutation*) to be a linear ordering $\sigma(1)\ldots\sigma(n)$ of $[n] = \{1,\ldots,n\}$, and identify σ with its *plot*, the set of points $\{(i,\sigma(i)) : 1 \leqslant i \leqslant n\}$ in the Euclidean plane.[1] We use $\mathcal{S}_n$ for the set of all n-permutations and $|\sigma|$ to denote the length of σ.

An *inversion* in σ is a pair i,j of indices such that $i < j$ and $\sigma(i) > \sigma(j)$, or equivalently two points in the plot of σ, one to the northwest of the other. The number of inversions in σ is denoted $\mathsf{inv}(\sigma)$, and we use $\mathcal{S}_{n,m} = \{\sigma \in \mathcal{S}_n : \mathsf{inv}(\sigma) = m\}$ for the set of all n-permutations with exactly m inversions. An n-permutation can have at most $\binom{n}{2}$ inversions; the *inversion density* of σ is defined to be the ratio $\rho_{\mathsf{inv}}(\sigma) = \mathsf{inv}(\sigma)/\binom{|\sigma|}{2}$.

We are interested in studying the properties of a typical large n-permutation with $m = m(n)$ inversions, for a given function $m(n)$, and we use $\boldsymbol{\sigma}_{n,m}$ to denote a permutation chosen uniformly at random from $\mathcal{S}_{n,m}$. Thus, $\boldsymbol{\sigma}_{n,m}$ can be seen as the natural analogue for permutations of the Erdős–Rényi random graph $\mathbf{G}_{n,m}$ [5]. The only prior work on $\boldsymbol{\sigma}_{n,m}$ of which we are aware is that of Acan and Pittel [1], who establish a sharp threshold for connectivity (that is, sum indecomposability) at $m = (6/\pi^2)n\log n$.

2010 Mathematics Subject Classification: 05A05, 05A16.

[†]Department of Computer and Information Sciences, University of Strathclyde, Glasgow, Scotland.

[1]For a very brief introduction to this perspective on permutations, see [2]; for more extended expositions, see either [3] or [7]; for an extensive recent survey, see [9].

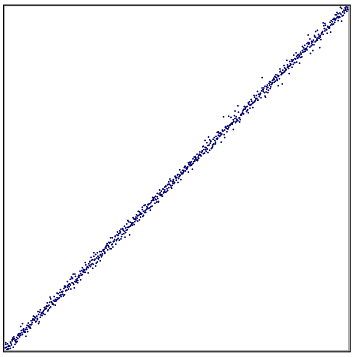

Figure 1: The plot of a randomly selected permutation on 825 points with inversion density 0.01.

In this paper, our focus is on the *local* structure of $\boldsymbol{\sigma}_{n,m}$ when m grows superlinearly but subquadratically with n, that is[2] when $n \ll m \ll n^2$. We call such (random) permutations *semi-sparse*.

Almost all the points of a semi-sparse permutation are close to the main diagonal in the following sense. The *absolute displacement* of the jth point of σ is $d_j(\sigma) = |\sigma(j) - j|$, its vertical distance from the main diagonal. Knuth [8] defined the *total displacement* $\mathsf{td}(\sigma) = \sum_{j=1}^{|\sigma|} d_j(\sigma)$ as a natural measure of how close σ is to the identity. Subsequently, Diaconis and Graham [4] proved that total displacement and number of inversions are related by the following double inequality: $\mathsf{inv}(\sigma) \leqslant \mathsf{td}(\sigma) \leqslant 2\mathsf{inv}(\sigma)$ for any permutation σ. Consequently, if $\boldsymbol{\sigma}_{n,m}$ is semi-sparse,

$$\lim_{n\to\infty} \mathbb{P}\big[d_{j_n}(\boldsymbol{\sigma}_{n,m}) \gg m/n\big] \;=\; 0,$$

for any sequence of positive integers (j_n) with $j_n \leqslant n$. Indeed, the analytic limit $\lim_{n\to\infty} \boldsymbol{\sigma}_{n,m}$ of semi-sparse permutations is the *permuton*[3] whose support is the main diagonal (see [6] and references therein for information about permutons). See Figure 1 for an illustration of a permutation with small inversion density chosen uniformly from $\mathcal{S}_{825,3399}$.

Unsurprisingly, then, if we pick two points randomly from a semi-sparse permutation $\boldsymbol{\sigma}_{n,m}$, then asymptotically almost surely[4] they do not form an inversion:

$$\text{If } \; i < j, \; \text{ then } \; \mathbb{P}\big[\boldsymbol{\sigma}_{n,m}(i) > \boldsymbol{\sigma}_{n,m}(j)\big] \;=\; \rho_{\mathsf{inv}}(\boldsymbol{\sigma}_{n,m}) \;=\; m/\tbinom{n}{2} \;\to\; 0 \; \text{ as } \; n \to \infty.$$

However, in stark contrast, the *local* structure is very different. Our primary result is that if we "zoom in" far enough, then locally a semi-sparse permutation is *uniform*, the restriction of $\boldsymbol{\sigma}_{n,m}$ to a sufficiently small interval being asymptotically uniformly distributed. The local structure of $\boldsymbol{\sigma}_{n,m}$ thus reveals nothing about its global form. As an illustration of this phenomenon, even for relatively small n, in the permutation in Figure 1 more than 47% of the pairs of *adjacent* points form inversions (that is, descents).

[2]We write $f(n) \ll g(n)$ or $g(n) \gg f(n)$ if $\lim_{n\to\infty} f(n)/g(n) = 0$, and write $f(n) \sim g(n)$ if $\lim_{n\to\infty} f(n)/g(n) = 1$.
[3]A permuton is a probability measure on the unit square with uniform marginals.
[4]A property $Q = Q(n)$ holds asymptotically almost surely if $\lim_{n\to\infty} \mathbb{P}[Q] = 1$.

1.1 Results

To state our results, we use the following notation and phraseology: $\sigma[i,j]$ denotes the sequence $\sigma(i)\sigma(i+1)\ldots\sigma(j)$, the restriction of σ to the interval $[i,j]$. We say that $\sigma[i,j]$ *forms* τ if the terms of $\sigma[i,j]$ are in the same relative order as those of the permutation τ, and we say that τ *occurs at position* j *in* σ if $\sigma[j, j+|\tau|-1]$ forms τ. In this context, the (*consecutive*) *subpermutation* τ is called a (*consecutive*) *pattern*.

Our first theorem establishes uniformity at a sufficiently small scale.

Theorem 1. *Suppose* $n \ll m \ll n^2/\log^2 n$ *and* $k \ll \sqrt{m/n}$. *Then, for any sequence of permutations* (τ_n) *with* $|\tau_n| = k$ *and any sequence of positive integers* (j_n) *with* $j_n \leqslant n+1-k$,

$$\mathbb{P}\big[\tau_n \textit{ occurs at position } j_n \textit{ in } \boldsymbol{\sigma}_{n,m}\big] \sim \frac{1}{k!}.$$

If we "zoom out" a little, then we have the following behaviour in the critical window, where the probability that $\boldsymbol{\sigma}_{n,m}$ locally looks like a permutation τ depends on τ's inversion density.

Theorem 2. *Suppose* $n \ll m \ll n^2/\log^2 n$ *and* $k \sim \alpha\sqrt{m/n}$ *for some* $\alpha > 0$. *Fix* $\rho \in [0,1]$. *Then, for any sequence of permutations* (τ_n) *with* $|\tau_n| = k$ *and* $\rho_{\mathsf{inv}}(\tau_n) \sim \rho$, *and any sequence of positive integers* (j_n) *with* $j_n \leqslant n+1-k$,

$$\mathbb{P}\big[\tau_n \textit{ occurs at position } j_n \textit{ in } \boldsymbol{\sigma}_{n,m}\big] \sim e^{(1-2\rho)\alpha^2/4}\frac{1}{k!}.$$

If our window is a bit wider, then any subpermutation with sufficient inversion density almost never occurs.

Theorem 3. *Suppose* $n \ll m \ll n^2/\log^2 n$ *and* $k \gg \sqrt{m/n}$. *Suppose* $m/nk^2 \ll \rho \leqslant 1$. *Then, for any sequence of permutations* (τ_n) *with* $|\tau_n| = k$ *and* $\rho_{\mathsf{inv}}(\tau_n) \sim \rho$, *and any sequence of positive integers* (j_n) *with* $j_n \leqslant n+1-k$,

$$\mathbb{P}\big[\tau_n \textit{ occurs at position } j_n \textit{ in } \boldsymbol{\sigma}_{n,m}\big] \ll \mathbb{P}\big[\boldsymbol{\sigma}_{n,m}[j_n, j_n+k-1] \textit{ is increasing}\big].$$

These first three theorems thus reveal the existence of a threshold at $k = \sqrt{m/n}$ for consecutive k-subpermutations of semi-sparse $\boldsymbol{\sigma}_{n,m}$ to be uniformly distributed.

If we turn our attention to pairs of points, the threshold for i, j being an inversion in $\boldsymbol{\sigma}_{n,m}$ is at the larger scale $j - i = m/n$. Below this, i, j is as likely to be an inversion as not, whereas above it, i, j is almost never an inversion.

Theorem 4. *Suppose* $n \ll m \ll n^2/\log^2 n$ *and* (j_n) *is any sequence of positive integers such that* $j_n \leqslant n-k$. *Then,*

$$\lim_{n\to\infty} \mathbb{P}\big[\boldsymbol{\sigma}_{n,m}(j_n) > \boldsymbol{\sigma}_{n,m}(j_n+k)\big] = \begin{cases} \frac{1}{2} & \textit{if } k \ll m/n, \\ 0 & \textit{if } k \gg m/n. \end{cases}$$

To conclude, we determine for $j - i$ in the critical window the exact asymptotic probability of i, j forming an inversion.

Theorem 5. *Suppose $n \ll m \ll n^2/\log^2 n$ and $k \sim \alpha m/n$ for some $\alpha > 0$. Then, for any sequence of positive integers (j_n) with $j_n \leqslant n - k$,*

$$\lim_{n\to\infty} \mathbb{P}\big[\boldsymbol{\sigma}_{n,m}(j_n) > \boldsymbol{\sigma}_{n,m}(j_n + k)\big] \;=\; \frac{e^\alpha(\alpha-1)+1}{(e^\alpha-1)^2} \;<\; \frac{1}{2}.$$

1.2 Methodology

The four essential components of our approach are as follows. Firstly, we establish a position independence result (Proposition 8): For any n and m, the random permutation $\boldsymbol{\sigma}_{n,m}$ "looks the same" in any two intervals of the same length. This means that we need only consider the structure of the first k points of $\boldsymbol{\sigma}_{n,m}$. Secondly, we represent permutations by their inversion sequences. Thirdly, we establish conditions under which we can use weak compositions to approximate large suffixes of inversion sequences of random semi-sparse permutations (Corollary 10). And, finally, we make use of a tripartition of these inversion sequences to enable us to apply these approximations to asymptotically enumerate certain classes of permutations (Proposition 11 and Proposition 14) which we use to yield our results.

In Section 2, we develop this framework, proving the position independence of consecutive patterns and establishing how to count semi-sparse permutations by approximating inversion sequences with weak compositions. Section 3 then contains the proofs of Theorems 1, 2 and 3, establishing the threshold for local uniformity. Finally, in Section 4, we prove Theorems 4 and 5, which establish the threshold for inversions.

The approximations we use are only valid when $m \ll n^2/\log^2 n$. (See the comment after the proof of Proposition 11.) For faster-growing m, a different approach is needed. Thus, the following questions remain open.

Question 6. *Suppose $m = \Omega(n^2/\log^2 n)$. How slowly does k need to grow so that for any sequence of permutations (τ_n) with $|\tau_n| = k$ and any sequence of positive integers (j_n) with $j_n \leqslant n+1-k$,*

$$\mathbb{P}\big[\tau_n \text{ occurs at position } j_n \text{ in } \boldsymbol{\sigma}_{n,m}\big] \;\sim\; \frac{1}{k!}?$$

Question 7. *Suppose $m = \Omega(n^2/\log^2 n)$. How slowly does k need to grow so that for any sequence of positive integers (j_n) with $j_n \leqslant n - k$,*

$$\lim_{n\to\infty} \mathbb{P}\big[\boldsymbol{\sigma}_{n,m}(j_n) > \boldsymbol{\sigma}_{n,m}(j_n + k)\big] \;=\; \tfrac{1}{2}?$$

It seems likely that techniques suitable for answering these questions for the remainder of the semi-sparse range ($m \ll n^2$) may well not be applicable to dense permutations, when $m \sim \rho\binom{n}{2}$. The only simple solution appears to be for Question 7 in the dense case when $\rho = \frac{1}{2}$, when the probability of $j, j+k$ forming an inversion is $1/2$ for all $k < n$.

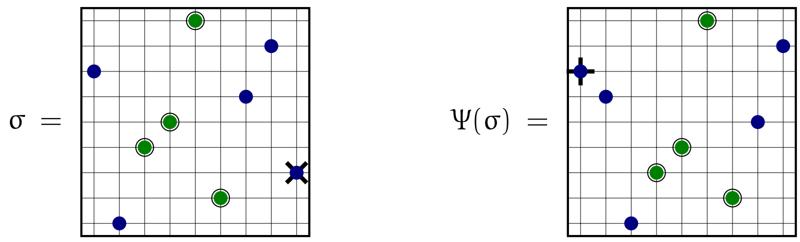

Figure 2: The bijection used in the proof of Proposition 8: the point marked **✕** is replaced by that marked **+**; the pattern 2341 occurs at position 3 in σ and at position 4 in $\Psi(\sigma)$

2 Foundations

In this section, we establish the basic framework we use to prove our results: the position independence of subpermutations and the asymptotic enumeration of inversion sequences by approximation using weak compositions.

2.1 Position independence

Our first observation is that the distribution of any consecutive pattern in $\boldsymbol{\sigma}_{n,m}$ is independent of its position. This holds for any given n and m. As a consequence, in subsequent arguments, we need only consider the occurrence of patterns at position 1 in $\boldsymbol{\sigma}_{n,m}$.

Proposition 8. *For any permutation $\tau \in \mathcal{S}_k$ and any positive $i, j \leqslant n+1-k$,*

$$\mathbb{P}\big[\tau \textit{ occurs at position } i \textit{ in } \boldsymbol{\sigma}_{n,m}\big] \;=\; \mathbb{P}\big[\tau \textit{ occurs at position } j \textit{ in } \boldsymbol{\sigma}_{n,m}\big].$$

This result follows from the existence of an operation that removes the last point from a permutation and adds a new first point in such a way as to preserve the number of inversions. This operation shifts patterns rightwards.

Proof. As illustrated in Figure 2, let $\Psi : \mathcal{S}_{n,m} \to \mathcal{S}_{n,m}$ be defined by

$$\Psi(\sigma) \;=\; \Psi(\sigma_1\sigma_2\ldots\sigma_n) \;=\; \sigma' \;=\; \sigma'_0\sigma'_1\ldots\sigma'_{n-1},$$

where $\sigma'_0 = n+1-\sigma_n$, and for $1 \leqslant i < n$,

$$\sigma'_i \;=\; \begin{cases} \sigma_i + 1, & \text{if } \sigma'_0 \leqslant \sigma_i < \sigma_n, \\ \sigma_i - 1, & \text{if } \sigma_n < \sigma_i \leqslant \sigma'_0, \\ \sigma_i, & \text{otherwise.} \end{cases}$$

Note that σ_n contributes $n - \sigma_n$ inversions to σ, and σ'_0 contributes the same number of inversions to σ'. For $0 < i < n$, the point σ'_i contributes the same number of inversions to σ' as σ_i does to σ. So $\mathsf{inv}(\sigma') = \mathsf{inv}(\sigma)$. Since Ψ preserves length and has a well-defined inverse, it is a bijection on $\mathcal{S}_{n,m}$.

If $\tau \in \mathcal{S}_k$ occurs at position $j \leqslant n-k$ in σ, then τ occurs at position $j+1$ in $\Psi(\sigma)$. Hence, if $1 \leqslant i, j \leqslant n+1-k$, then τ occurs at position i in σ if and only if τ occurs at position j in $\Psi^{j-i}(\sigma)$. □

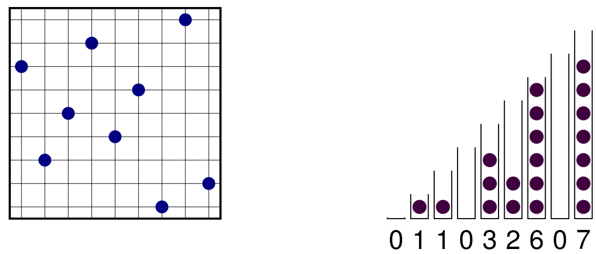

Figure 3: A permutation with 20 inversions and its inversion sequence

2.2 Inversion sequences and weak compositions

Key to our analysis is the representation of permutations as *inversion sequences*. The inversion sequence of an n-permutation σ is $(e_j)_{j=1}^{n}$, where $e_j = \left|\{i : i < j \text{ and } \sigma_i > \sigma_j\}\right|$ is the number of inversions of σ whose right end is at position j. Note that $\mathsf{inv}(\sigma) = \sum_j e_j$.

Clearly, for each j, we have $e_j < j$, and in fact sequences satisfying this condition whose sum equals m are in bijection with n-permutations having m inversions. Each e_j can be considered to be the number of *balls* in a *box* whose capacity is $j-1$. This is illustrated in Figure 3. Rather than working directly with permutations, we investigate the properties of this balls-in-boxes model when the number of balls (m) is superlinear but subquadratic in the number of boxes (n).

This analysis is aided by using (unrestricted) *weak compositions* to approximate inversion sequences. A *weak* t*-composition of* s is a sequence of t non-negative integers whose sum is s. In terms of our balls-in-boxes model, we have s balls in t boxes, each of whose capacity is *unlimited*. We use $\mathcal{C}_{t,s}$ to denote the set of all weak t-compositions of s. Clearly, the number of such compositions is given by $|\mathcal{C}_{t,s}| = \binom{s+t-1}{s}$.

The foundation for our approximation is the fact that asymptotically almost surely, no term in a semi-sparse weak t-composition of s exceeds its expected value of s/t by a factor greater than $\log t$. A specific instance of this proposition is proved by Acan and Pittel in [1]; we generalise their proof.

Proposition 9. *Suppose* $t \ll s \ll t^2/\log t$ *and* χ_t *is chosen uniformly at random from* $\mathcal{C}_{t,s}$*. Then, for any* $\varepsilon > 0$ *and sufficiently large* t,

$$\mathbb{P}\left[\textit{some term of } \chi_t \textit{ is at least } (1+\varepsilon)\tfrac{s}{t}\log t\right] \;\leqslant\; t^{-\varepsilon/2}.$$

Proof. Let L_r be the number of terms of χ_t whose value is at least r. By the first moment method and linearity of expectation, $\mathbb{P}[L_r > 0] \leqslant \mathbb{E}[L_r] = t\mathbb{P}[x_1 \geqslant r]$, where x_1 is the first term of χ_t.

Now, for any positive r,

$$\mathbb{P}[x_1 \geqslant r] \;=\; \frac{\sum_{j=0}^{s-r} |\mathcal{C}_{t-1,j}|}{|\mathcal{C}_{t,s}|} \;=\; \frac{s!(s+t-r-1)!}{(s-r)!(s+t-1)!} \;\leqslant\; \left(\frac{s}{s+t-r}\right)^r \;=\; \left(1-\frac{t-r}{s+t-r}\right)^r.$$

Thus, $\mathbb{P}[L_r > 0] \leqslant t\left(1-\frac{t-r}{s+t-r}\right)^r \leqslant t\exp\left(-\frac{r(t-r)}{s+t-r}\right)$.

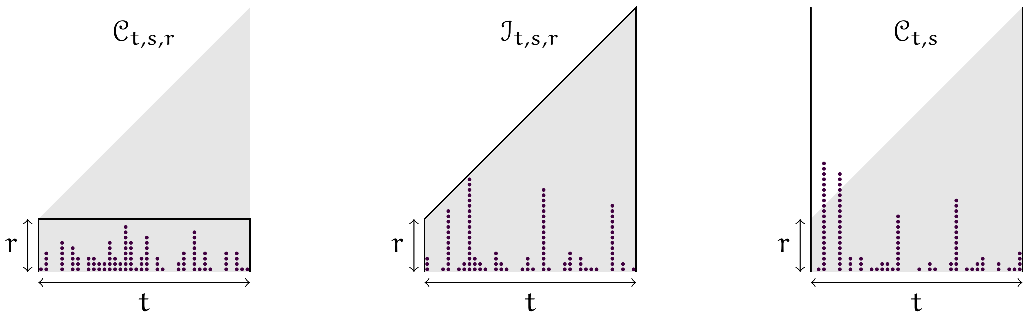

Figure 4: A restricted weak composition, an inversion sequence suffix, and an unrestricted weak composition

Suppose $r = \alpha\frac{s}{t}\log t$, where $\alpha = 1+\varepsilon$. By the bound on s, we have $r < t$ for sufficiently large t. Rearrangement then yields

$$\frac{r(t-r)}{s+t-r} \;=\; \left(\frac{1-\frac{\alpha s \log t}{t^2}}{1+\frac{t}{s}-\frac{\alpha \log t}{t}}\right)\alpha \log t \;\geqslant\; \left(\alpha-\tfrac{\varepsilon}{2}\right)\log t \;=\; \left(1+\tfrac{\varepsilon}{2}\right)\log t$$

for sufficiently large t, as long as $t \ll s \ll t^2/\log t$.

Hence, for t large enough, $\mathbb{P}\big[L_r > 0\big] \;\leqslant\; t\,e^{-(1+\varepsilon/2)\log t} \;=\; t^{-\varepsilon/2}$. □

We use weak compositions to approximate all but the first few terms in inversion sequences. Let $\mathcal{I}_{t,s,r}$ denote the set of *inversion sequence suffixes* that consists of weak t-compositions $(e_j)_{j=1}^{t}$ of s in which $e_j < j + r$ for each j. See the middle of Figure 4 for an illustration.

If r is sufficiently large, then the number of these inversion sequence suffixes may be approximated by the number of unrestricted weak compositions.

Corollary 10. *Suppose* $t \ll s \ll t^2/\log t$ *and* $r \geqslant (1+\varepsilon)\frac{s}{t}\log t$ *for some positive* ε*. Then*

$$|\mathcal{I}_{t,s,r}| \;\sim\; \binom{s+t-1}{s}.$$

Proof. Let $\mathcal{C}_{t,s,r}$ be the set of *restricted* weak t-compositions of s, in which every term is less than r. Clearly $\mathcal{C}_{t,s,r} \subset \mathcal{I}_{t,s,r} \subset \mathcal{C}_{t,s}$ (see Figure 4). By Proposition 9, we have $|\mathcal{C}_{t,s,r}| \sim |\mathcal{C}_{t,s}|$ under the specified conditions on t, s and r. So, $|\mathcal{I}_{t,s,r}| \sim |\mathcal{C}_{t,s}| = \binom{s+t-1}{s}$. □

2.3 Counting permutations

To make use of this approximation, we partition the terms of the inversion sequence of an n-permutation into three parts. Given some $k > 0$ and $r \geqslant k$, part $\mathcal{A}$ consists of the first k terms of the sequence (the first k boxes), $\mathcal{B}$ consists of the next $r-k$ terms, and $\mathcal{C}$ consists of the remaining $n-r$ terms. See Figure 5 for an illustration.

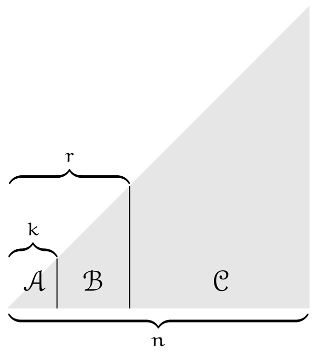

Figure 5: The partitioning of inversion sequences

We use this tripartition as follows: Firstly, we place a specific pattern of length k in part $\mathcal{A}$. Secondly, the value of r is chosen so that we can approximate the number of ways of filling part $\mathcal{C}$ by using Corollary 10. Finally, we sum over each possible way of placing balls in part $\mathcal{B}$.

Suppose we decide to place exactly ℓ balls in the first k boxes (part $\mathcal{A}$) in some particular way. Let $N^{k,\ell}_{n,m} = |\mathcal{I}_{n-k,m-\ell,k}|$ be the number of ways of distributing an additional $m-\ell$ balls among the boxes in parts $\mathcal{B}$ and $\mathcal{C}$.

Equivalently, $N^{k,\ell}_{n,m}$ is the number of n-permutations with m inversions whose first k points form some particular permutation having ℓ inversions. For example, $N^{k,0}_{n,m}$ is the number of n-permutations with m inversions whose first k points are increasing. These have inversion sequences that begin with k zeros (the first k boxes are empty).

If $B = \binom{r}{2} - \binom{k}{2}$ denotes the total capacity of part $\mathcal{B}$ (boxes $k+1, \ldots, r$), and $b_i = |\mathcal{I}_{r-k,i,k}|$ is the number of distinct ways of placing exactly i balls in these $r-k$ boxes (for $i = 0, \ldots, B$), then we can express $N^{k,\ell}_{n,m}$ as follows:

$$N^{k,\ell}_{n,m} \;=\; \sum_{i=0}^{B} b_i \left|\mathcal{I}_{n-r,m-\ell-i,r}\right|, \tag{1}$$

where we sum over the possible choices for the contents of part $\mathcal{B}$.

We now approximate the terms in this sum by using Corollary 10.

Proposition 11. *If* $n \ll m \ll n^2/\log^2 n$ *and* $r = \left\lceil \frac{2m}{n} \log n \right\rceil$, *then*

$$N^{k,\ell}_{n,m} \;\sim\; \sum_{i=0}^{B} b_i \binom{m-i \,+\, n-r-1 \,-\, \ell}{n-r-1},$$

where $B = \binom{r}{2} - \binom{k}{2}$ *and* $b_i = |\mathcal{I}_{r-k,i,k}|$.

Proof. To use Corollary 10 to approximate $|\mathcal{I}_{n-r,m-\ell-i,r}|$ for any nonnegative $\ell \leqslant \binom{k}{2}$ and $i \leqslant B$, we require the following three inequalities to hold for some $\varepsilon > 0$:

$$r \;\geqslant\; (1+\varepsilon)\frac{m-(\ell+i)}{n-r}\log(n-r) \qquad \text{and} \qquad n-r \;\ll\; m-(\ell+i) \;\ll\; \frac{(n-r)^2}{\log(n-r)}.$$

Since $m \ll n^2/\log n$, we have $r = \lceil \frac{2m}{n}\log n \rceil \ll n$, and thus the first inequality is satisfied for sufficiently large n.

Similarly, the third inequality holds as a consequence of $m \ll n^2/\log n$ and $r \ll n$.

Finally, given that $m \ll n^2/\log^2 n$, we have

$$r^2 \;=\; \left\lceil \frac{2m}{n}\log n \right\rceil^2 \;\ll\; \frac{2m\log n}{n}\,\frac{2n}{\log n} \;=\; 4m.$$

Thus, $m \gg \binom{r}{2}$, which is the maximum possible value of $\ell + i$. Together with $n \ll m$, this is sufficient to ensure that the second inequality is satisfied.

The result then follows from (1). $\square$

Note that the proof of this proposition requires $m \ll n^2/\log^2 n$. For faster-growing m, one or more of the three inequalities fails to hold, so Corollary 10 cannot be applied.

Our final goal in this section is to compare values of $N^{k,\ell}_{n,m}$ for different ranges of values for ℓ. We make use of the following two simple results.

Proposition 12. *If* $x \ll y$, *then*

$$\lim_{x\to\infty} \binom{y}{x}\Big/\binom{y-\delta}{x} \;=\; \begin{cases} 1 & \textit{if } \delta \ll y/x, \\ e^{\alpha} & \textit{if } \delta \sim \alpha y/x, \textit{ for any } \alpha > 0, \\ \infty & \textit{if } \delta \gg y/x. \end{cases}$$

Proof. By Stirling's approximation,

$$\tbinom{y}{x}\Big/\tbinom{y-\delta}{x} \;\sim\; \left(1 - \tfrac{x\delta}{(y-x)(y-\delta)}\right)^{y+\frac{1}{2}} \left(1 + \tfrac{\delta}{y-x-\delta}\right)^{x} \left(1 + \tfrac{x}{y-x-\delta}\right)^{\delta},$$

from which the result can be seen to follow. $\square$

Proposition 13. *Suppose that we have positive* a_i, x_i, y_i *for* $i = 1, \ldots, n$ *and that there are* L, U *such that* $L \leqslant x_i/y_i \leqslant U$ *for each* i. *If* $X = \sum_{i=1}^{n} a_i x_i$ *and* $Y = \sum_{i=1}^{n} a_i y_i$, *then* $L \leqslant X/Y \leqslant U$.

Proof.

$$YL \;=\; \sum_{i=1}^{n} a_i y_i L \;\leqslant\; \sum_{i=1}^{n} a_i x_i \;=\; X \;=\; \sum_{i=1}^{n} a_i x_i \;\leqslant\; \sum_{i=1}^{n} a_i y_i U \;=\; YU. \qquad \square$$

Recall that $N^{k,\ell}_{n,m}$ is the number of n-permutations with m inversions whose first k points form some particular permutation having ℓ inversions. The following proposition establishes the threshold for change in the asymptotic value of $N^{k,\ell}_{n,m}$ for semi-sparse permutations in terms of the growth of ℓ with respect to n and m.

Proposition 14. *Suppose* $n \ll m \ll n^2/\log^2 n$. *Then,*

$$\begin{aligned} N_{n,m}^{k,\ell} &\sim N_{n,m}^{k,0} && \textit{if } \ell \ll m/n, \\ N_{n,m}^{k,\ell} &\sim e^{-\beta} N_{n,m}^{k,0} && \textit{if } \ell \sim \beta m/n, \\ N_{n,m}^{k,\ell} &\ll N_{n,m}^{k,0} && \textit{if } \ell \gg m/n. \end{aligned}$$

Proof. Suppose $r \ll n$ and $i \ll m$. Then $\frac{m-i+n-r-1}{n-r-1} \sim \frac{m}{n}$. Set $r = \left\lceil \frac{2m}{n} \log n \right\rceil$.

Suppose $\ell \ll m/n$. Then, by Proposition 12, we have $\binom{m-i+n-r-1-\ell}{n-r-1} \sim \binom{m-i+n-r-1}{n-r-1}$, and so, by Propositions 11 and 13, $N_{n,m}^{k,\ell} \sim N_{n,m}^{k,0}$.

Similarly, if $\ell \sim \beta m/n$ then $\binom{m-i+n-r-1-\ell}{n-r-1} \sim e^{-\beta} \binom{m-i+n-r-1}{n-r-1}$, and thus $N_{n,m}^{k,\ell} \sim e^{-\beta} N_{n,m}^{k,0}$.

Finally, suppose $\ell \gg m/n$. Then, by Proposition 12, $\binom{m-i+n-r-1-\ell}{n-r-1} \ll \binom{m-i+n-r-1}{n-r-1}$. Hence, by Propositions 11 and 13, $N_{n,m}^{k,\ell} \ll N_{n,m}^{k,0}$. □

3 Threshold for local uniformity

We are now in a position to establish the threshold for local uniformity. First we prove that a semi-sparse permutation $\boldsymbol{\sigma}_{n,m}$ is indeed locally uniform. This is Theorem 1, which we restate here.

Theorem 1. *Suppose* $n \ll m \ll n^2/\log^2 n$ *and* $k \ll \sqrt{m/n}$. *Then, for any sequence of permutations* (τ_n) *with* $|\tau_n| = k$ *and any sequence of positive integers* (j_n) *with* $j_n \leqslant n + 1 - k$,

$$\mathbb{P}\big[\tau_n \textit{ occurs at position } j_n \textit{ in } \boldsymbol{\sigma}_{n,m}\big] \sim \frac{1}{k!}.$$

Proof. Since $|\tau_n| = k \ll \sqrt{m/n}$, we have $\mathsf{inv}(\tau_n) \leqslant \binom{k}{2} \ll m/n$. Therefore, by Proposition 14, we have $N_{n,m}^{k,\mathsf{inv}(\tau_n)} \sim N_{n,m}^{k,0}$, and hence

$$\mathbb{P}\big[\boldsymbol{\sigma}_{n,m}[1,k] \text{ forms } \tau_n\big] \sim \mathbb{P}\big[\boldsymbol{\sigma}_{n,m}[1,k] \text{ is increasing}\big],$$

and this is true whatever sequence (τ_n) is selected. Thus, asymptotically, every possible choice for the pattern formed by the first k points of $\boldsymbol{\sigma}_{n,m}$ is equally probable, and so

$$\mathbb{P}\big[\tau_n \text{ occurs at position 1 in } \boldsymbol{\sigma}_{n,m}\big] \sim \frac{1}{k!}.$$

The result then follows from the position independence of the distribution of consecutive patterns in $\boldsymbol{\sigma}_{n,m}$ (Proposition 8). □

We now consider the behaviour in the critical window. In order to do this we require tail bounds on the distribution of the number of inversions in a random n-permutation. We use $\boldsymbol{\sigma}_n$ to denote a permutation chosen uniformly from $\mathcal{S}_n$.

Proposition 15. *For any $\theta > 0$,*

$$\mathbb{P}\big[\big|\rho_{\mathsf{inv}}(\boldsymbol{\sigma}_n) - \tfrac{1}{2}\big| \;>\; \theta\big] \;<\; 2e^{-\theta^2 n}.$$

Proof. We apply Hoeffding's inequality, which states that if $X_1, \ldots, X_n$ are independent random variables such that $0 \leqslant X_i \leqslant u_i$ for all i, and $S_n = \sum_{i=1}^{n} X_i$, then

$$\mathbb{P}\big[\big|S_n - \mathbb{E}[S_n]\big| \;>\; t\big] \;<\; 2\exp\!\left(-\frac{2t^2}{\sum_{i=1}^{n} u_i^2}\right).$$

Now, as illustrated by the balls-in-boxes model, $\mathsf{inv}(\boldsymbol{\sigma}_n) \sim \sum_{i=1}^{n} \mathsf{Unif}[0, i-1]$, the sum of n independent discrete uniform random variables, so we can set $u_i = i - 1$, yielding

$$\sum_{i=1}^{n} u_i^2 \;=\; \frac{(2n-1)n(n-1)}{6},$$

which does not exceed $\frac{2}{n}\binom{n}{2}^2$, from which the result follows directly. $\square$

In the critical window, where $k \sim \alpha\sqrt{m/n}$, the asymptotic probability of a particular consecutive pattern depends (only) on its inversion density.

Theorem 2. *Suppose $n \ll m \ll n^2/\log^2 n$ and $k \sim \alpha\sqrt{m/n}$ for some $\alpha > 0$. Fix $\rho \in [0,1]$. Then, for any sequence of permutations (τ_n) with $|\tau_n| = k$ and $\rho_{\mathsf{inv}}(\tau_n) \sim \rho$, and any sequence of positive integers (j_n) with $j_n \leqslant n + 1 - k$,*

$$\mathbb{P}\big[\tau_n \textit{ occurs at position } j_n \textit{ in } \boldsymbol{\sigma}_{n,m}\big] \;\sim\; e^{(1-2\rho)\alpha^2/4}\frac{1}{k!}.$$

Proof. Since $|\tau_n| = k \sim \alpha\sqrt{m/n}$ and $\mathsf{inv}(\tau_n) \sim \rho\binom{k}{2}$, we have $\mathsf{inv}(\tau_n) \sim \frac{\rho\alpha^2 m}{2n}$. Therefore, by Proposition 14, we have $N_{n,m}^{k,\mathsf{inv}(\tau_n)} \sim e^{-\rho\alpha^2/2} N_{n,m}^{k,0}$, and hence

$$\mathbb{P}\big[\boldsymbol{\sigma}_{n,m}[1,k] \text{ forms } \tau_n\big] \;\sim\; e^{-\rho\alpha^2/2}\,\mathbb{P}\big[\boldsymbol{\sigma}_{n,m}[1,k] \text{ is increasing}\big]. \tag{2}$$

Now clearly, for any given k, we have $\sum_{\varphi\in[0,1]} \mathbb{P}\big[\rho_{\mathsf{inv}}(\boldsymbol{\sigma}_{n,m}[1,k]) = \varphi\big] = 1$, where the sum should be understood to be over the finite set of possible inversion densities of k-permutations. Equivalently,

$$\sum_{\varphi\in[0,1]} \Big|\mathcal{S}_{k,\varphi\binom{k}{2}}\Big|\,\mathbb{P}\big[\boldsymbol{\sigma}_{n,m}[1,k] \text{ forms } \pi_k^{\varphi}\big] \;=\; 1,$$

where, for each valid value of φ, we choose π_k^{φ} to be some k-permutation with inversion density exactly φ.

There are exactly $k!\,\mathbb{P}\big[\rho_{\mathsf{inv}}(\boldsymbol{\sigma}_k) = \varphi\big]$ permutations of length k with inversion density φ. So,

$$k! \sum_{\varphi\in[0,1]} \mathbb{P}\big[\rho_{\mathsf{inv}}(\boldsymbol{\sigma}_k) = \varphi\big]\,\mathbb{P}\big[\boldsymbol{\sigma}_{n,m}[1,k] \text{ forms } \pi_k^{\varphi}\big] \;=\; 1.$$

Now, in an analogous manner to (2), for every valid φ we have

$$\mathbb{P}\big[\boldsymbol{\sigma}_{n,m}[1,k] \text{ forms } \pi_k^{\varphi}\big] \;\sim\; e^{-\varphi\alpha^2/2}\,\mathbb{P}\big[\boldsymbol{\sigma}_{n,m}[1,k] \text{ is increasing}\big],$$

where we take limits over those n for which $\varphi\binom{k}{2} \in \mathbb{N}$. Thus,

$$k!\,\mathbb{P}\big[\boldsymbol{\sigma}_{n,m}[1,k] \text{ is increasing}\big] \sum_{\varphi\in[0,1]} \mathbb{P}\big[\rho_{\mathsf{inv}}(\boldsymbol{\sigma}_k) = \varphi\big] e^{-\varphi\alpha^2/2} \;\sim\; 1. \tag{3}$$

We now make use of our tail bounds for the inversion density. By Proposition 15,

$$\mathbb{P}\big[\big|\rho_{\mathsf{inv}}(\boldsymbol{\sigma}_k) - \tfrac{1}{2}\big| \;>\; k^{-1/4}\big] \;<\; 2e^{-\sqrt{k}},$$

and $e^{-\varphi\alpha^2/2}$ is no greater than 1 for any φ. So the contribution to the sum in (3) from inversion densities that differ from ½ by more than $k^{-1/4}$ is less than $2e^{-\sqrt{k}}$, which tends to zero.

On the other hand, the remaining contribution to the sum (from inversion densities close to ½) lies between

$$\big(1-2e^{-\sqrt{k}}\big)e^{-(1+k^{-1/4})\alpha^2/4} \quad\text{and}\quad e^{-(1-k^{-1/4})\alpha^2/4},$$

and so tends to $e^{-\alpha^2/4}$. Thus, $k!\,\mathbb{P}\big[\boldsymbol{\sigma}_{n,m}[1,k] \text{ is increasing}\big] \sim e^{\alpha^2/4}$.

Applying (2) then yields

$$\mathbb{P}\big[\tau_n \text{ occurs at position 1 in } \boldsymbol{\sigma}_{n,m}\big] \;\sim\; e^{(1-2\rho)\alpha^2/4}\frac{1}{k!},$$

and the result follows from the position independence of the distribution of consecutive patterns in $\boldsymbol{\sigma}_{n,m}$ (Proposition 8). $\square$

Any larger subpermutation with a sufficient number of inversions almost never occurs.

Theorem 3. *Suppose $n \ll m \ll n^2/\log^2 n$ and $k \gg \sqrt{m/n}$. Suppose $m/nk^2 \ll \rho \leqslant 1$. Then, for any sequence of permutations (τ_n) with $|\tau_n| = k$ and $\rho_{\mathsf{inv}}(\tau_n) \sim \rho$, and any sequence of positive integers (j_n) with $j_n \leqslant n+1-k$,*

$$\mathbb{P}\big[\tau_n \textit{ occurs at position } j_n \textit{ in } \boldsymbol{\sigma}_{n,m}\big] \;\ll\; \mathbb{P}\big[\boldsymbol{\sigma}_{n,m}[j_n, j_n+k-1] \textit{ is increasing}\big].$$

Proof. Since $|\tau_n| = k \gg \sqrt{m/n}$ and $\mathsf{inv}(\tau_n) \sim \rho\binom{k}{2}$, with $\rho \gg m/nk^2$, we have $\mathsf{inv}(\tau_n) \gg m/n$. Therefore, by Proposition 14, we have $N_{n,m}^{k,\mathsf{inv}(\tau_n)} \ll N_{n,m}^{k,0}$, and hence

$$\mathbb{P}\big[\boldsymbol{\sigma}_{n,m}[1,k] \text{ forms } \tau_n\big] \;\ll\; \mathbb{P}\big[\boldsymbol{\sigma}_{n,m}[1,k] \text{ is increasing}\big].$$

The result then follows from the position independence of the distribution of consecutive patterns in $\boldsymbol{\sigma}_{n,m}$ (Proposition 8). $\square$

Theorems 1, 2 and 3 thus reveal the existence of a threshold at $k = \sqrt{m/n}$ for the uniform distribution of consecutive k-subpermutations of semi-sparse $\boldsymbol{\sigma}_{n,m}$.

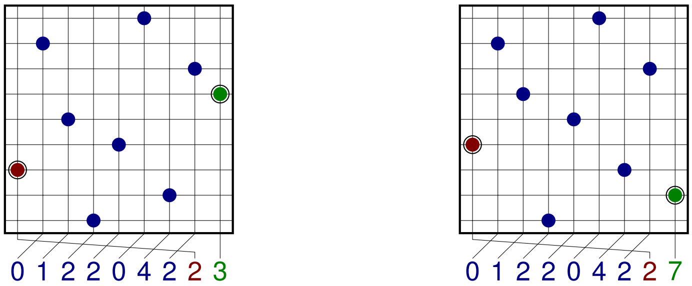

Figure 6: Permutations built by adjoining new initial and final points, together with their nonstandard inversion sequences

4 Threshold for inversions

We now turn our attention to the uniformity of *inversions*: How close do indices i and j need to be for i, j to be as likely to form an inversion in $\boldsymbol{\sigma}_{n,m}$ as not? We begin with some counting.

Suppose π is a $(k-1)$-permutation. Let us consider how new first and last points may be adjoined to π so that the resulting permutation has exactly ℓ more inversions than π. Specifically, we want to determine how many distinct ways there are to construct a $(k+1)$-permutation τ such that $\tau[2,k]$ forms π and $\mathsf{inv}(\tau) - \mathsf{inv}(\pi) = \ell$. The answer depends on k and ℓ, and on whether or not $\tau(1)\tau(k+1)$ forms an inversion. It doesn't depend on any other properties of π or τ.

Suppose $0 \leqslant \ell \leqslant k-1$. In this case, $\mathsf{inv}(\tau) - \mathsf{inv}(\pi) = \ell$ precisely when $\tau(k+1) - \tau(1) = k - \ell$. For example, on the left of Figure 6, we have $\tau(9) - \tau(1) = 3$ with $k = 8$ and $\ell = 5$. Note that $\tau(1)\tau(k+1)$ does not form an inversion. In this case it is readily checked that there are exactly $\ell + 1$ distinct ways to construct an appropriate τ from any given π, adding ℓ inversions.

On the other hand, if $k \leqslant \ell \leqslant 2k-1$, then $\mathsf{inv}(\tau) - \mathsf{inv}(\pi) = \ell$ whenever $\tau(1) - \tau(k+1) = \ell + 1 - k$. For example, on the right of Figure 6, we have $\tau(1) - \tau(9) = 2$ with $k = 8$ and $\ell = 9$. Note that, in this case, $\tau(1)\tau(k+1)$ does form an inversion. It can be seen that there are exactly $2k - \ell$ distinct ways to build a suitable τ from any given $(k-1)$-permutation π, adding ℓ inversions.

We use nonstandard inversion sequences to represent permutations built this way, reflecting how they are constructed. Specifically, we represent a $(k+1)$-permutation τ constructed from a $(k-1)$-permutation π by a sequence $(e_j)_{j=1}^{k+1}$, where the first $k-1$ terms form the standard inversion sequence for π, so

$$e_j = \left|\{i : 1 \leqslant i < j \text{ and } \tau(i+1) > \tau(j+1)\}\right|$$

for $1 \leqslant j \leqslant k-1$, and the final two terms are given by

$$\begin{aligned} e_k &= \left|\{i : 2 \leqslant i \leqslant k \text{ and } \tau(1) > \tau(i)\}\right|, \\ e_{k+1} &= \left|\{i : 1 \leqslant i < k+1 \text{ and } \tau(i) > \tau(k+1)\}\right|, \end{aligned}$$

recording the number of inversions created by adjoining the new initial and final points, respectively. See Figure 6 for two examples. Note that we still have $e_j < j$ for each j.

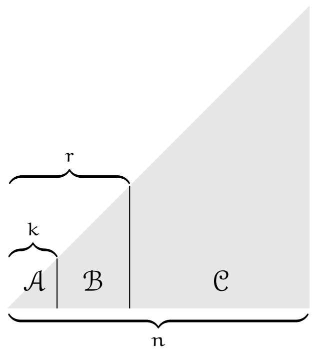

Figure 7: The partitioning of modified inversion sequences

We now construct a modified inversion sequence for each n-permutation σ, in which the first $k+1$ terms form the nonstandard inversion sequence for $\sigma[1, k+1]$ and subsequent terms are standard. Thus, the kth and $(k+1)$th terms record the number of inversions created by adjoining $\sigma(1)$ and $\sigma(k+1)$ to $\sigma[2, k]$.

As before, for a given $r \geqslant k$, we partition the terms (or boxes) of modified inversion sequences into three parts (see Figure 7). Let $A = \binom{r}{2} - (2k-1)$ be the total capacity of parts $\mathcal{A}$ and $\mathcal{B}$ excluding the "special" boxes k and $k+1$. That is, A is the capacity of boxes $1, \dots, k-1$ and $k+2, \dots, r$. Now, for each $i = 0, \dots, A$, let a_i be the number of distinct ways of placing exactly i balls in these $r-2$ boxes.

Then the number of n-permutations σ with m inversions in which $\sigma(1) < \sigma(k+1)$ is given by

$$N^{k\nearrow}_{n,m} = \sum_{i=0}^{A} a_i \sum_{\ell=0}^{k-1} (\ell+1) \left|\mathcal{I}_{n-r,m-\ell-i,r}\right|, \tag{4}$$

where $\ell+1$ is the number of ways of choosing $\sigma(1)$ and $\sigma(k+1)$, or equivalently the contents of boxes k and $k+1$, so as to contribute ℓ inversions.

Similarly, the number of n-permutations σ with m inversions in which $\sigma(1)$ and $\sigma(k+1)$ form an inversion is given by

$$\begin{aligned} N^{k\searrow}_{n,m} &= \sum_{i=0}^{A} a_i \sum_{\ell=k}^{2k-1} (2k-\ell) \left|\mathcal{I}_{n-r,m-\ell-i,r}\right| \\ &= \sum_{i=0}^{A} a_i \sum_{\ell=0}^{k-1} (\ell+1) \left|\mathcal{I}_{n-r,m-\ell-i-(2k-2\ell-1),r}\right|, \end{aligned} \tag{5}$$

where the second expression results from the change of variable $\ell = 2k-1-\ell$.

As we did in Proposition 11, we now use Corollary 10 to approximate the terms of these sums.

Proposition 16. *If* $n \ll m \ll n^2/\log^2 n$ *and* $r = \lceil \frac{2m}{n} \log n \rceil$*, then*

$$N^{k\nearrow}_{n,m} \sim \sum_{i=0}^{A} a_i \sum_{\ell=0}^{k-1} (\ell+1) \binom{m-\ell-i+n-r-1}{n-r-1},$$

$$N^{k\searrow}_{n,m} \sim \sum_{i=0}^{A} a_i \sum_{\ell=0}^{k-1} (\ell+1) \binom{m-\ell-i+n-r-1-(2k-2\ell-1)}{n-r-1},$$

where $A = \binom{r}{2} - (2k-1)$ *and* a_i *is as defined above.*

Proof. Reasoning almost identical to that in Proposition 11 shows that the choice of r and the bounds on n and m are sufficient to ensure that Corollary 10 can be used to approximate $|\mathcal{I}_{n-r,m-\ell-i,r}|$ and $|\mathcal{I}_{n-r,m-\ell-i-(2k-2\ell-1),r}|$ for any nonnegative $i \leqslant A$ and $\ell < k$. The result then follows from (4) and (5). $\square$

Using these approximations, we can now exhibit a threshold at $k = m/n$ for the uniformity of inversions.

Theorem 4. *Suppose* $n \ll m \ll n^2/\log^2 n$ *and* (j_n) *is any sequence of positive integers such that* $j_n \leqslant n-k$*. Then,*

$$\lim_{n\to\infty} \mathbb{P}\big[\boldsymbol{\sigma}_{n,m}(j_n) > \boldsymbol{\sigma}_{n,m}(j_n+k)\big] = \begin{cases} \frac{1}{2} & \textit{if } k \ll m/n, \\ 0 & \textit{if } k \gg m/n. \end{cases}$$

Proof. We compare $N^{k\searrow}_{n,m}$ against $N^{k\nearrow}_{n,m}$.

As long as $r \ll n$ and $\ell < k$, by Proposition 12,

$$\lim_{n\to\infty} \binom{m-\ell-i+n-r-1-(2k-2\ell-1)}{n-r-1} \Big/ \binom{m-\ell-i+n-r-1}{n-r-1} = \begin{cases} 1 & \text{if } k \ll m/n, \\ 0 & \text{if } k \gg m/n. \end{cases}$$

Set $r = \lceil \frac{2m}{n} \log n \rceil$. If $k \ll m/n$, then by Propositions 16 and 13, we have $N^{k\searrow}_{n,m} \sim N^{k\nearrow}_{n,m}$, and so

$$\mathbb{P}\big[\boldsymbol{\sigma}_{n,m}(1) > \boldsymbol{\sigma}_{n,m}(k+1)\big] \sim \mathbb{P}\big[\boldsymbol{\sigma}_{n,m}(1) < \boldsymbol{\sigma}_{n,m}(k+1)\big] \sim \tfrac{1}{2}.$$

On the other hand, if $k \gg m/n$, then $N^{k\searrow}_{n,m} \ll N^{k\nearrow}_{n,m}$, and so $\mathbb{P}\big[\boldsymbol{\sigma}_{n,m}(1) < \boldsymbol{\sigma}_{n,m}(k+1)\big] \sim 1$.

The result then follows from the position independence of the distribution of consecutive patterns in $\boldsymbol{\sigma}_{n,m}$ (Proposition 8). $\square$

In the critical window, we have the following behaviour.

Theorem 5. *Suppose* $n \ll m \ll n^2/\log^2 n$ *and* $k \sim \alpha m/n$ *for some* $\alpha > 0$*. Then, for any sequence of positive integers* (j_n) *with* $j_n \leqslant n-k$*,*

$$\lim_{n\to\infty} \mathbb{P}\big[\boldsymbol{\sigma}_{n,m}(j_n) > \boldsymbol{\sigma}_{n,m}(j_n+k)\big] = \frac{e^{\alpha}(\alpha-1)+1}{(e^{\alpha}-1)^2} < \frac{1}{2}.$$

Proof. It can be checked (using a computer algebra package or otherwise) that

$$S_{\nearrow} \;=\; \sum_{\ell=0}^{k-1}(\ell+1)\binom{y-\ell}{x} \;=\; \frac{(y+1)(y+2)\binom{y}{x}-(y+1-k)(y+2+xk+k)\binom{y-k}{x}}{(x+1)(x+2)},$$

and

$$\begin{aligned} S_{\searrow} \;&=\; \sum_{\ell=0}^{k-1}(\ell+1)\binom{y-\ell-(2k-2\ell-1)}{x} \\ &=\; \frac{(y-x-2k)(y+1-x-2k)\binom{y+1-2k}{x}-(y+1-x-k)(y-xk-x-3k)\binom{y+1-k}{x}}{(x+1)(x+2)}. \end{aligned}$$

Thus, if $k \sim \alpha y/x$ and $x \ll y$, then by Proposition 12,

$$\begin{aligned} S_{\nearrow} \;&\sim\; \frac{y^2}{x^2}\binom{y}{x}\big(1-(1+\alpha)e^{-\alpha}\big), \\ S_{\searrow} \;&\sim\; \frac{y^2}{x^2}\binom{y}{x}\big(e^{-2\alpha}-(1-\alpha)e^{-\alpha}\big). \end{aligned}$$

So, if we let $x = n-r-1$ and $y = m-i+x$, by Propositions 16 and 13 we have the following for $p_\alpha = \mathbb{P}\big[\boldsymbol{\sigma}_{n,m}(1) > \boldsymbol{\sigma}_{n,m}(k+1)\big]$, the probability that $1, k+1$ forms an inversion in $\boldsymbol{\sigma}_{n,m}$:

$$p_\alpha \;=\; \frac{N^{k\searrow}_{n,m}}{N^{k\nearrow}_{n,m} \;+\; N^{k\searrow}_{n,m}} \;\sim\; \frac{e^{-2\alpha}-(1-\alpha)e^{-\alpha}}{1-(1+\alpha)e^{-\alpha} \;+\; e^{-2\alpha}-(1-\alpha)e^{-\alpha}} \;=\; \frac{e^{\alpha}(\alpha-1)+1}{(e^{\alpha}-1)^2}.$$

For $\alpha > 0$, this probability decreases as α increases, since

$$\frac{dp_\alpha}{d\alpha} \;=\; -\frac{e^{\alpha}\left(e^{\alpha}(\alpha-2)+\alpha+2\right)}{(e^{\alpha}-1)^3}$$

and

$$e^{\alpha}(\alpha-2)+\alpha+2 \;=\; \sum_{n\geqslant 3}(n-2)\alpha^n/n!,$$

each term in the Taylor expansion being positive. Thus, for positive α, we have $p_\alpha < \frac{1}{2}$, since $\lim_{\alpha\to 0} p_\alpha = \frac{1}{2}$.

The result then follows from the position independence of the distribution of consecutive patterns in $\boldsymbol{\sigma}_{n,m}$ (Proposition 8). □

Acknowledgements

The catalyst for this work was a brief conversation with Jakub Sliačan at the 26th British Combinatorial Conference in July 2017 at the University of Strathclyde in Glasgow. I am also grateful to Thomas Selig for acting as a sounding-board for some of my initial ideas on this and related topics. *Mathematica* [10] was used for experimentation and algebraic manipulation.

Soli Deo gloria!

References

[1] Hüseyin Acan and Boris Pittel. On the connected components of a random permutation graph with a given number of edges. *J. Combin. Theory Ser. A*, 120(8):1947–1975, 2013.

[2] David Bevan. Permutation patterns: basic definitions and notation. *arXiv:1506.06673*, 2015.

[3] Miklós Bóna. *Combinatorics of Permutations*. CRC Press, second edition, 2012.

[4] Persi Diaconis and R. L. Graham. Spearman's footrule as a measure of disarray. *J. Royal Statist. Soc. Ser. B*, 39:262–268, 1977.

[5] P. Erdős and A. Rényi. On the evolution of random graphs. *Magyar Tud. Akad. Mat. Kutató Int. Kőzl.*, 5:17–61, 1960.

[6] Richard Kenyon, Daniel Král', Charles Radin, and Peter Winkler. Permutations with fixed pattern densities. *Random Structures & Algorithms*, 2019.

[7] Sergey Kitaev. *Patterns in Permutations and Words*. Springer, 2011.

[8] Donald E. Knuth. *The Art of Computer Programming. Vol. 3: Sorting and Searching*. Addison-Wesley, 1973.

[9] Vincent Vatter. Permutation classes. In Miklós Bóna, editor, *The Handbook of Enumerative Combinatorics*. CRC Press, 2015.

[10] Wolfram Research, Inc. Mathematica. wolfram.com/mathematica.